He visited Catalonia many times to speak at congresses and conferences, as the author of books, as a member of doctoral thesis juries, etc.

Here he would find Albert Dou whom he so admired, some of his disciples, many lecturers with whom he collaborated and many friends and followers. His last visits were for the tribute in Girona to our own Lluís A. Santaló and for the inauguration at IEC headquarters of the programme for promoting mathematical talent —a programme he had led successfully in Madrid and which is now taking off in Catalonia.

Heaven has a new light. It is a star-shaped polyhedron and anyone looking at it will discover things there. It is him.

Thank you, Miguel Guzmán, for your example. We will never forget you.

Claudi Alsina i Català
UPC

Problem Section

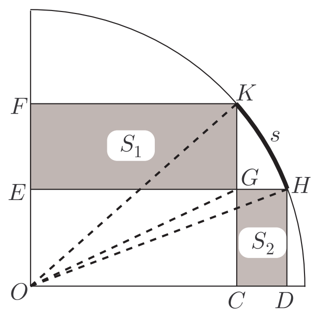
From the first issues of *SCM/Notícies*, one of the most popular sections of our Newsletter has been the Problem Section, where our readers can find and post mathematical questions and problems being either curious or interesting (or both). And, of course, they can also propose solutions to the problems announced in previous issues. For each issue, the editorial board chooses and publishes the nicest problems and the best solutions. As a sample, we reproduce three of the problems that appeared in this section.

Selection of posted problems with the corresponding solutions:

A52 *SCM/Notícies* 16, December 2001. (59th Annual William Lowell Putnam Mathematical Competition)

Let s be an arbitrary arc of the unit circle, situated at the first quadrant. Let A be the area of the trapezoidal region defined between the arc and the "x" axis, and let B be the area of the trapezoidal region defined between the arc and the "y" axis. Show that $A + B$ only depends on the length of the arc s and not on its position.

Solution: (Redaction)



Let S_1 be the area of the rectangle $EGKF$, and S_2 be the area of the rectangle $CDHG$. It is clear that $S_1 = 2 \cdot \text{Area}\,(\triangle OGK)$ because the triangle and the rectangle have the same basis and height. By the same reason, $S_2 = 2 \cdot \text{Area}\,(\triangle OGH)$. Hence,

$$\begin{aligned} A + B &= S_1 + S_2 + 2 \cdot \text{Area}\,(GHK) = \\ &= 2 \cdot (\text{Area}\,(\triangle OGK) \\ &\quad + \text{Area}\,(\triangle OGH) + \text{Area}\,(GHK)) = \\ &= 2 \cdot (\text{Area Sector}\,(OHK)) = \\ &= \text{Length of arc}\,(s). \end{aligned}$$

A59 *SCM/Notícies* 18, January 2003. (A german suggestion for an International Mathematical Olympiad)

Let a, b and m be integral numbers such that

$$\frac{a^2 + b^2}{ab + 1} = m \geq 0.$$

Show that, m is a square.

Solution: (Solution by Carles Romero, IES "Manuel Blancafort", La Garriga)

i) If one of the numbers is zero, the proposition is trivial.

ii) Assume $a=b$. Then we have:

$$0 \leq m = \frac{2a^2}{a^2+1} \Longrightarrow 2a^2 = ma^2 + m$$
$$\Longrightarrow (2-m)a^2 = m \geq 0.$$

The only possibilities are $m=0$ and $m=1$, both being squares. The corresponding values for a are $a=0$ and $a=\pm 1$, respectively.

iii) It is not possible that $a \neq 0$ and $b \neq 0$ have different signs, because $a^2+b^2>0$ and so is $ab+1>0$, hence $ab \geq 0$.

iv) Let us assume now $0<a<b$. In this case,

$$b^2 - amb + a^2 - m = 0$$

is a degree two equation with respect to the unknown "b", having the given value of b as a solution. Let b' be the other solution (possibly equal to b). They satisfy

$$b + b' = am\,; \quad bb' = a^2 - m.$$

This is saying that if a and b produce the integer $m>0$, then $b' = am - b$ and a also do. Hence, by (iii), $b' \geq 0$. Additionally, using $a<b$, we deduce that

$$ab' \leq bb' = a^2 - m < a^2 \quad \Longrightarrow \quad b' < a.$$

Summarizing, if $0<a<b$ produce the integer $m>0$, then $0 \leq b' < a$ also do.

v) Repeating the previous process several times, we obtain a strictly descending list of integral numbers, no two consecutive ones being of different sign. This implies that the list contains the zero. And immediately next to it there is the positive integer $\sqrt{m} = \text{m.c.d.}(a,b)$.

A63 *SCM/Notícies* 20, November 2004. (Proposed by Pelegrí Viader, Universitat Pompeu Fabra, Barcelona)

Let $f:[0,1] \to \mathbb{R}$ be a continuous function such that $f(0)=f(1)$. Show that:

a) For every positive integer n there exists a *horizontal cord* of the graph of f having length $1/n$ (a *horizontal cord* is a segment between two points in the graph of f having the same second coordinate).

b) f does not necessarily have horizontal cords of length not being the inverse of a positive integer.

(*Theorem of the universal cord*)

Solution: (Solution by Albert Ferreiro Castilla, student)

The case $n=1$ is trivial, by the hypothesis of the problem.

Suppose now $n>1$. The problem consists on finding a pair of points, $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$, in such a way that $|x_1 - x_2| = \dfrac{1}{n}$ and $f(x_1) = f(x_2)$. To show this, let us consider the following set of points in the real plain:

$$\mathcal{C}_n = \left\{ \left(x, x + \frac{1}{n} \right) \mid 0 \leq x \leq 1 - \frac{1}{n} \right\} \subset \mathbb{R}^2,$$

which is nothing else but the set of all pairs of real numbers between 0 and 1, and being exactly $\dfrac{1}{n}$ apart from each other (and ordered, first the small then the big one). Let us observe that this set $\mathcal{C}_n \subset \mathbb{R}^2$ is a segment of a straight line in the plane. Let us consider the following map, which is clearly continuous:

$$F : \mathcal{C}_n \to \mathbb{R}$$
$$F(x,y) = f(x) - f(y).$$

If we find $(x,y) \in \mathcal{C}_n$ with $F(x,y)=0$, here we have the pair of points generating the cord of length $\dfrac{1}{n}$, and we are done. If we prove that, on the set $\mathcal{C}_n$, the map F takes simultaneously positive and negative values, then Bolzano's theorem will ensure us the existence of a zero, and we are also done. Let us assume that F has no zero (and so, does not take simultaneously positive and negative values), and we will find a contradiction.

Pick the pair $\left(0, \dfrac{1}{n}\right)$ from the set $\mathcal{C}_n$, and let us apply F. Since $F\left(0, \dfrac{1}{n}\right) \neq 0$, we can assume without lost of generality, that $F\left(0, \dfrac{1}{n}\right) > 0$.

Let us consider now the pair $\left(\dfrac{1}{n}, \dfrac{2}{n}\right)$: again by our assumption,

$$F\left(\frac{1}{n}, \frac{2}{n}\right) > 0.$$

Repeating the same argument, we can conclude that

$$F\left(\frac{i-1}{n}, \frac{i}{n}\right) > 0\,, \qquad i = 1, \dots, n.$$

Putting all these inequalities together, we have:

$$F\left(0, \frac{1}{n}\right) > 0 \Rightarrow f(0) > f\left(\frac{1}{n}\right)$$
$$F\left(\frac{1}{n}, \frac{2}{n}\right) > 0 \Rightarrow f\left(\frac{1}{n}\right) > f\left(\frac{2}{n}\right)$$
$$\dots$$
$$F\left(\frac{n-1}{n}, \frac{n}{n}\right) > 0 \Rightarrow$$
$$f\left(\frac{n-1}{n}\right) > f\left(\frac{n}{n}\right) = f(1)$$

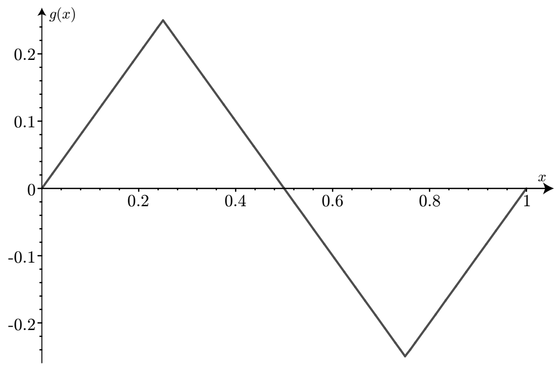
But then, $f(0) > f(1)$, which contradicts the hypothesis of the problem. Consequently, the map F is negative or zero on some the pairs in $\mathcal{C}_n$. Thus, we have the result. We remark the fact that we have been looking for points whose distance to each other is the inverse of a positive integer; this is essential sence, otherwise, the list of points obtained in the argument would not end at 1, which is what we need to contradict the hypothesis of the problem. That is, for lengths different from $1/n$ the given argument is no longer working.

Let us consider now the second part of the problem. The claim is obvious for the inverse of every number between 0 and 1: it is clear that no cord can exist with length $\frac{1}{0.5} = 2$ and this is not the interesting case. We shall construct an example concerning the inverse of a number bigger than 1:

Consider the following continuous function on the interval $[0, 1]$:

$$g(x) = \begin{cases} x\,, & \text{si } 0 \le x \le \frac{1}{4} \\ -x + \frac{1}{2}\,, & \text{si } \frac{1}{4} \le x \le \frac{3}{4} \\ x - 1\,, & \text{si } \frac{3}{4} \le x \le 1 \end{cases}$$

It clearly satisfies $g(0) = g(1) = 0$. So the hypothesis hold, but there are no cords of length bigger than $\frac{1}{2}$ (see the graph of g).



Mathematics PhD theses in Catalonia

Mathematics PhD theses in Catalonia

Three of the Catalan universities offer Mathematics courses, both at undergraduate and at doctorate level. The Faculty of Mathematics of University of Barcelona, the Faculty of Sciences of the Autonomous University of Barcelona and the Faculty of Mathematics and Statistics of the Technical University of Catalonia all have their corresponding doctoral programmes in Mathematics, in which doctors receive ongoing training in the different areas and specialities represented in Catalonia.

The quality of these doctoral programmes has increased considerably in recent years and is currently comparable to that of many of the best European and American universities. The outgoing students are fully initiated and ready to carry out research in their corresponding specialities, as shown by the fact that many of them participate actively in research groups both here and abroad and successfully devote themselves to mathematical research. Furthermore, following the trend of the most prestigious universities, there is a growing interest on the part of companies in the technology and financial sectors in hiring increasing numbers of doctors in Mathematics in order to develop their more specialised work. This is a clear indication of the vitality of our three doctoral programmes and of the usefulness of high-level Mathematics, beyond the realm of pure research.

To give an idea of the subjects dealt with